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LETTER TO THE EDITOR

Bistable oscillator driven by two periodic fields

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Abstract

Two periodic fields acting on a bistable oscillator are assumed to have different timescales, which allows the exclusion of one of these fields via the separation of timescales. The second field is taken into account by harmonic linearization, which finally leads to a linear equation. Dynamic stabilization of the unstable point and a vibrational resonance follow from simple calculations.

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Landa and McClintock [1] recently performed a numerical analysis of a bistable oscillator subject to two periodic fields and found a phenomenon similar to stochastic resonance, which they called vibrational resonance. The purpose of this Letter is to give a simple analytical treatment of this and some related phenomena. We consider the bistable oscillator, which is described by the following equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}x}{\mathrm{d}t} - \omega_0^2 x + \beta x^3 = A\sin(\omega t) + C\sin(\Omega t)\,. \tag{1}$$

Let us suppose that one of the fields has an amplitude larger than the barrier height, $C > \frac{\omega_0^2}{4\beta}$, and high frequency, $\Omega \gg \omega$. The former means that this field during each half-period transfers the system from one potential well to the other. A similar situation holds in a random system where the large-amplitude field is replaced by a random force, which plays the same role of switching a system between two minima. Therefore, by choosing an appropriate relation between the input signal $A\sin(\omega t)$ and the amplitude C of the large signal (or the strength of the noise) one can obtain a non-monotonic dependence of the output signal on the amplitude C (vibrational resonance [1]) or on the noise strength (stochastic resonance [2]).

Consider first the case $A = 0$. We look for the solution of equation (1) in the form

$$x(t) = y(t) - \frac{C\sin(\Omega t)}{\Omega^2}. \tag{2}$$

The first term on the right-hand side will be assumed to vary significantly only over times of the order of t, while the second term varies rapidly. Substituting (2) into (1) with $A = 0$, one can perform an averaging over a single cycle time of $\sin(\Omega t)$. All odd powers of $\sin(\Omega t)$

vanish under the average while the $\sin^2(\Omega t)$ term will give $\frac{1}{2}$. Finally, one obtains the following equation for $X(t)$, the mean value of $y(t)$ during the oscillation period, $X(t) = \langle y(t) \rangle$:

$$\frac{d^2X}{dt^2} + \alpha \frac{dX}{dt} - \left(\omega_0^2 - \frac{3\beta C^2}{2\Omega^4}\right) X + \beta X^3 = 0. \tag{3}$$

For $\frac{3\beta C^2}{2\Omega^4} \geqslant \omega_0^2$ the phenomenon of dynamic stabilization [3] occurs, namely the high-frequency external field transforms the previously unstable position $X = 0$ into a stable one. This phenomenon has been found numerically for the pendulum [4], and by an electronic analogue experiment for the Duffing oscillator [3]. Notice that in contrast to the well known phenomenon of a stabilization of the reverse pendulum by high-frequency parametric oscillations [5], here the high-frequency oscillations enter the equation of motion additively and not multiplicatively.

Returning to equation (1) with $A \neq 0$, we notice that the slowly varying term $A\sin(\omega t)$ does not change under the averaging over the short time, and equation (1) can now be rewritten as

$$\frac{d^2X}{dt^2} + \alpha \frac{dX}{dt} - \left(\omega_0^2 - \frac{3\beta C^2}{2\Omega^4}\right) X + \beta X^3 = A\sin(\omega t). \tag{4}$$

One can say that equation (4) is the 'coarse-grained' (with respect to time) version of equation (1).

In order to solve equation (4) one needs some additional assumptions, which could be weak, fast or slow driving [6]. We restrict ourselves to the simplest treatment of a resonance in a nonlinear oscillator [5], used, in particular, for random and periodic forces [7]. The approximate solution of equation (4) can be written as

$$X(t) \approx \Theta \sin(\omega t - \theta). \tag{5}$$

Retaining only the first term in a Fourier series of the nonlinear term in equation (4) and averaging it over the period $\frac{2\pi}{\omega}$ of the external field, one can replace the βX^3 term by $\frac{3\beta\Theta^2}{4} X$. Then equation (4) reduces to

$$\frac{d^2X}{dt^2} + \alpha \frac{dX}{dt} + \varpi^2 X = A\sin(\omega t) \tag{6}$$

with the renormalized frequency

$$\varpi = \left(\frac{3\beta\Theta^2}{4} + \frac{3\beta C^2}{2\Omega^4} - \omega_0^2\right)^{\frac{1}{2}}. \tag{7}$$

Equation (4) is identically satisfied by (5) if

$$\alpha\omega\Theta = A\sin\theta \qquad \frac{3\beta\Theta^3}{4} - \left(\omega_0^2 - \frac{3\beta C^2}{2\Omega^4} + \omega^2\right)\Theta = A\cos\theta. \tag{8}$$

Excluding Θ from equations (7) and (8), one obtains

$$\varpi^2 = \frac{3\beta C^2}{2\Omega^4} - \omega_0^2 + \frac{3\beta A^2}{4\left[\alpha^2\omega^2 + \left(\varpi^2 - \omega^2\right)^2\right]}. \tag{9}$$

A resonance in the linear equation (6) occurs when $\omega = \varpi$. Substituting the latter in equation (9), one can find the relation between the amplitudes and frequencies of the two driving fields in equation (1) which produce the resonant behaviour. This condition has the form

$$\omega^2 = \frac{3\beta C^2}{2\Omega^4} + \frac{3\beta A^2}{4\alpha^2\omega^2} - \omega_0^2 \tag{10}$$

and is of course model dependent. According to our model, two periodic forces acting on a bistable nonlinear oscillator transfer it into a single-well linear oscillator, and equation (10) determines the condition of the dynamic stabilization of the previously unstable point $x = 0$.

In conclusion, we have shown that an additional periodic field is able not only to control chaos in nonlinear systems [8], but also to have control over the shift of the resonance frequency (equation (10)), which is desirable in systems of practical importance such as electronic devices or lasers.

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